

# PURELY EXPONENTIAL GROWTH OF CUSP-UNIFORM ACTIONS

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**ABSTRACT.** Suppose that a countable group  $G$  admits a cusp-uniform action on a hyperbolic space  $(X, d)$  such that  $G$  is of divergent type. The main result of the paper is characterizing the purely exponential growth type of the orbit growth function by a condition introduced by Dal'bo-Otal-Peigné. For geometrically finite Cartan-Hadamard manifolds with pinched negative curvature this condition ensures the finiteness of Bowen-Margulis-Sullivan measures. In this case, our result recovers a theorem of Roblin (in a weaker form). Our main tool is the Patterson-Sullivan measures on the Gromov boundary of  $X$ , and a variant of the Sullivan shadow lemma called partial shadow lemma. This allows us to prove that the purely exponential growth of either cones, or partial cones or horoballs is also equivalent to the condition of Dal'bo-Otal-Peigné. These results are further used in the paper [21].

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## 1. INTRODUCTION

Suppose that a group  $G$  admits a proper and isometric action on a proper hyperbolic space  $(X, d)$  such that  $G$  does not fix a point in the Gromov boundary  $\partial X$ . We are interested in studying the asymptotic feature of the  $G$ -orbits in  $X$  via the Patterson-Sullivan measure on the boundary  $\partial X$ . This is a recurring scheme in the setting of simply connected Cartan-Hadamard manifolds with pinched negative curvature. The novelty of the present paper is the generality of the results obtained for a kind of cusp-uniform actions we explain now.

The *limit set*  $\Lambda(G)$  of  $G$  is the set of accumulation points in  $\partial X$  of a  $G$ -orbit in  $X$ . It is well-known that  $G$  acts minimally on  $\Lambda(G)$  as a convergence group action. A point  $p \in \Lambda(G)$  is called *parabolic* if the stabilizer  $G_p = \{g \in G : gp = p\}$  is infinite so that its limit set is  $\{p\}$ . Denote by  $\mathcal{P}$  the set of maximal parabolic subgroups in

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2000 *Mathematics Subject Classification.* Primary 20F65, 20F67.

*Key words and phrases.* Purely exponential growth, relatively hyperbolic groups, cusp-uniform actions, Patterson-Sullivan measures.

$G$ . Let  $\mathcal{C}(\Lambda G)$  be the *convex-hull* of  $\Lambda(G)$  such that it is the union of all bi-infinite geodesics with both endpoints in  $\Lambda(G)$ . It is known that  $\mathcal{C}(\Lambda G)$  is a  $G$ -invariant quasi-convex subset. So  $\mathcal{C}(\Lambda G)$  is itself a hyperbolic space.

We consider the following definition stated in [11], generalizing the definition of a geometrically finite Kleinian group in [12].

**Definition 1.1** (Cusp-uniform action). Assume that there is a  $G$ -invariant system of (open) horoballs  $\mathbb{U}$  centered at parabolic points of  $G$  such that the action of  $G$  on the complement

$$\mathcal{C}(\Lambda G) \setminus \mathcal{U}, \mathcal{U} := \cup_{U \in \mathbb{U}} U$$

is co-compact. Then the pair  $(G, \mathcal{P})$  is said to be *relatively hyperbolic*, and the action of  $G$  on  $X$  is called *cuspidal-uniform*.

*Remark.* We emphasize that  $G$  is not required to be finitely generated. Since  $\partial X$  is metrizable, the group  $G$  could be at most countable [9, Corollary 7.1]. The free product of two non-finitely generated countable groups gives rise to a non-finitely generated cuspidal-uniform action.

*Remark.* We do not request that  $\partial X = \Lambda G$  and consider using the convex hull of  $\Lambda G$  instead. This allows us to include the natural examples of a Kleinian group acting on  $\mathbb{H}^n$  ( $n \geq 2$ ) with  $\Lambda G \not\subseteq \mathbb{S}^{n-1}$ .

In fact, a group (pair) being relatively hyperbolic admits many equivalent formulations, for instance [7], [3], [6], [14], [11] and [9]. Since we are interested in the asymptotic growth of the orbits of a cuspidal-uniform action, the notion of a critical exponent shall be our focus.

Choose a basepoint  $o \in X$ . Set  $N(o, R) := \{g \in G : d(o, go) \leq R\}$ . Consider the Poincaré series for a subset  $\Gamma \subset G$ :

$$\mathcal{P}_\Gamma(s, o) = \sum_{g \in \Gamma} \exp(-sd(o, go)), \quad s \geq 0.$$

Note that the *critical exponent* of  $\mathcal{P}_G(s, o)$  is given by

$$\delta_\Gamma = \limsup_{R \rightarrow \infty} \frac{\log \#(N(o, R) \cap \Gamma)}{R},$$

which is independent of the choice of  $o \in X$ . Observe that  $\mathcal{P}_\Gamma(s, o)$  diverges for  $s < \delta_\Gamma$  and converges for  $s > \delta_\Gamma$ .

The group  $G$  is of *divergent type* (resp. *convergent type*) (with respect to the action of  $G$  on  $X$ ) if  $\mathcal{P}_G(s, o)$  is divergent (resp. convergent) at  $s = \delta_G$ . Note that whether  $\mathcal{P}_G(s, o)$  is of divergent type does not depend on the choice of  $o$ .

**1.1. Patterson-Sullivan measures and the partial shadow lemma.** The main tool in the paper is the Patterson-Sullivan measures (PS-measures for shorthand) on the Gromov boundary of  $X$ . For discrete groups acting on  $n$ -dimensional hyperbolic spaces ( $n \geq 2$ ) the theory of PS-measures was established by Patterson [15] for  $n = 2$  and generalized by Sullivan [17] in all dimensions. Furthermore, Sullivan gave a way to construct a flow-invariant measure on the unit tangent bundle for hyperbolic  $n$ -manifolds, which turns out to be the Bowen-Margulis measures studied earlier. Sullivan's construction is very robust and applies in rather general settings, for instance in CAT(-1) spaces in [16]. Following [16] we call this measure Bowen-Margulis-Sullivan measure (BMS measure for short). If  $X = \mathbb{H}^n$  ( $n \geq 2$ ) is a real hyperbolic space, then the BMS measure is finite [18, Theorem 3]. However,

for Cartan-Hadamard manifolds with pinched negative curvature, it was observed in [24] that the finiteness of the BMS measure depends crucially on the geometry on cusps.

In what follows, we shall discuss in details three conditions with increasing generalities on parabolic subgroups.

By definition, a cusp-uniform action of  $G$  on  $X$  has a *parabolic gap property* (PGP) if  $\delta_G > \delta_P$  for every maximal parabolic subgroup  $P$ . This property was introduced by Dal'bo-Otal-Peigné in [5] to deduce that  $G$  is of divergent type. Furthermore, the following condition was introduced to ensure the finiteness of the BMS measure for Cartan-Hadamard manifolds with pinched negative curvature. We put their condition in the setting of cusp-uniform actions.

**Definition 1.2.** Let  $G$  admit a cusp-uniform action as above such that  $\delta_G < \infty$ . Then  $G$  satisfies a *Dal'bo-Otal-Peigné (DOP) condition* if the following

$$(1) \quad \sum_{p \in P} d(o, po) \exp(-\delta_G d(o, po)) < \infty$$

holds for every  $P \in \mathcal{P}$  and  $o \in X$ .

*Remark.* The condition (1) depends only on conjugacy classes of  $P \in \mathcal{P}$ . In practice, it suffices to verify this condition for finitely many conjugacy classes in  $\mathcal{P}$ . Note also that (1) does not depend on the choice of  $o \in X$ .

**Theorem 1.3.** [5] *Let  $G$  admit a cusp-uniform action on a simply connected Cartan-Hadamard manifold  $X$  with pinched negative curvature. Suppose that  $G$  is of divergent type. Then the BMS measure is finite on the unit tangent bundle of  $X/G$  if and only if  $G$  satisfies the DOP condition.*

It is obvious that if  $G$  has the parabolic gap property then  $G$  satisfies the DOP condition. They imply the third one - a *parabolic convergence property* (PCP):

$$(2) \quad \sum_{p \in P} \exp(-\delta_G d(o, po)) < \infty$$

for every  $P \in \mathcal{P}$  and  $o \in X$ . This property turns out to be always true: it is trivial for the convergent case; see Lemma 3.9 for the divergent case. In fact, the PCP property is a crucial fact used to establish the partial shadow lemma below.

The above three conditions on parabolic groups will be important for further discussions.

In a greater generality, Coornaert [4] has established the theory of PS-measures on the limit set of a discrete group acting on a hyperbolic space. Assuming that  $G$  is of divergent type, we first generalize Dal'bo-Otal-Peigné's results to the setting of cusp-uniform actions.

**Theorem 1.4** (=Proposition 3.15). *Let  $G$  admit a cusp-uniform action on  $X$  such that  $\delta_G < \infty$  and  $G$  is of divergent type. Then the Patterson-Sullivan measure  $\{\mu_v\}_{v \in G}$  on  $\partial X$  is a quasi-conformal density without atoms. Moreover,  $\{\mu_v\}_{v \in G}$  is unique and ergodic.*

*Remark.* The statement that  $\{\mu_v\}_{v \in G}$  on  $\partial X$  is a quasi-conformal density was proved by Coornaert [4].

In the theory of PS-measures, a key tool is the Sullivan Shadow Lemma, which connects the geometry inside and the measure on boundary. We shall prove a

variant of the Shadow Lemma that holds for *partial shadows*. Before stating the lemma lets introduce a technical definition.

**Definition 1.5.** For a path  $p$  in  $X$ , a point  $v \in p$  is called an  $(\epsilon, R)$ -*transition* point for  $\epsilon, R \geq 0$  if the  $R$ -neighborhood  $p \cap B(v, R)$  around  $v$  in  $p$  is not contained in the  $\epsilon$ -neighborhood of any  $U \in \mathbb{U}$ .

*Remark.* The notion of transition points was due to Hruska [11] in the setting of Cayley graphs.

Hence, the *partial shadow*  $\Pi_{r,\epsilon,R}(go)$  at  $g$  for  $r \geq 0$  is the set of boundary points  $\xi \in \Lambda G$  such that some geodesic  $[o, \xi]$  intersects  $B(go, r)$  and contains an  $(\epsilon, R)$ -transition point  $v$  in  $B(go, 2R)$ .

**Lemma 1.6** (=Partial Shadow Lemma 3.18). *Under the assumption of Theorem 1.4, there are constants  $r_0, \epsilon, R \geq 0$  such that the following holds*

$$\exp(-\delta_G d(o, go)) < \mu_1(\Pi_{r,\epsilon,R}(go)) <_r \exp(-\delta_G d(o, go)),$$

for any  $g \in G$  and  $r \geq r_0$ .

**1.2. Characterizing the purely exponential growth type.** This is the main contribution of this paper. The idea of using PS-measures to study growth problems goes back to works of Patterson [15] and Sullivan [17], see [17, Theorem 9] for example. We now introduce several growth functions associated to orbits and horoballs.

For  $\Delta \geq 0, n \geq 0$ , consider the orbit in an annulus

$$(3) \quad A(go, n, \Delta) := \{h \in G : n - \Delta \leq d(o, ho) - d(o, go) < n + \Delta\},$$

for any  $g \in G$ .

The  $r$ -*cone*  $\Omega_r(go)$  at  $go$  for  $r \geq 0$  is the set of elements  $h \in G$  such that some geodesic  $[o, ho]$  intersects  $B(go, r)$ . For  $r, \epsilon, R > 0$ , the notion of an  $(\epsilon, R)$ -*partial  $r$ -cone*  $\Omega_{r,\epsilon,R}(go)$  at  $go$  is defined similarly, by demanding the existence of  $(\epsilon, R)$ -transition points on  $[o, ho]$   $2R$ -close to  $go$ . See Section 3 for precise definitions.

Consider the orbit in a cone:

$$(4) \quad \Omega_r(go, n, \Delta) := \Omega_r(go) \cap A(go, n, \Delta)$$

and in a partial cone:

$$(5) \quad \Omega_{r,\epsilon,R}(go, n, \Delta) := \Omega_{r,\epsilon,R}(go) \cap A(go, n, \Delta),$$

for any  $g \in G, n \geq 0$ .

We say that  $G$  has *purely exponential orbit growth* if there exists  $\Delta > 0$  such that

$$(6) \quad \# A(go, n, \Delta) \asymp \exp(\delta_G n)$$

for  $n \geq 1$ . If there exist  $r, \epsilon, R, \Delta$  such that (6) holds for  $\# \Omega_r(go, n, \Delta)$  (resp.  $\# \Omega_{r,\epsilon,R}(go, n, \Delta)$ ) then  $G$  has *purely exponential orbit growth in cones* (resp. *partial cones*).

The main result of the paper is that the above growth functions of purely exponential type are all equivalent to the DOP condition. They are in fact also equivalent to the purely exponential growth of horoballs defined as follows.

Let  $\mathbb{U}$  be the collection of horoballs in definition 1.1 of a cusp-uniform action. Consider

$$(7) \quad H(o, n, \Delta) := \{U \in \mathbb{U} : |d(o, U) - n| \leq \Delta\}.$$

It is clear that the equivalent type of the horoballs growth function  $\sharp H(o, n, \Delta)$  does not depend on the choice of  $o$  and  $\mathbb{U}$ . We say  $G$  has *purely exponential horoball growth* if

$$\sharp H(o, n, \Delta) \asymp \exp(\delta_G n)$$

for some  $\Delta > 0$ .

So our main theorem reads.

**Theorem 1.7.** *Suppose  $G$  admits a cusp-uniform action on a proper hyperbolic space  $(X, d)$  such that  $\delta_G < \infty$  and  $G$  is of divergent type. Then the following statements are equivalent:*

- (1)  $G$  satisfies the DOP condition.
- (2)  $G$  has the purely exponential orbit growth.
- (3)  $G$  has the purely exponential orbit growth in (partial) cones.
- (4)  $G$  has the purely exponential horoball growth.

*Remark.* (on the proof) It is possible that  $\delta_G = \infty$ , see Example 1 in [8, Section 3.4]. The direction “(4)  $\Rightarrow$  (3)” is trivial by definition of cusp-uniform actions; “(3)  $\Rightarrow$  (4)” is proved in Section 4 by using the PCP property. The main direction is to prove “(1)  $\Rightarrow$  (2)” in Section 5, along the way the converse direction could be easier established. “(2)  $\Rightarrow$  (3)” is due to the partial shadow lemma.

The following corollary follows from Theorem 1.3, recovering a result of Roblin in the setting of CAT(-1) spaces.

**Corollary 1.8.** [16, Théorème 4.1] *Suppose that  $G$  admits a cusp-uniform action on a simply connected Riemannian manifold with pinched negative curvature such that  $G$  is of divergent type. Then  $G$  has finite BMS measure if and only if the orbit growth function of  $G$  is purely exponential.*

We record the following corollary of Theorem 1.7.

**Corollary 1.9.** *Suppose that  $G$  satisfies the DOP condition. There exist  $r, \epsilon, R, \Delta > 0$  such that the following holds for any  $n \geq 0$ .*

- (1)  $\sharp A(o, n, \Delta) \asymp \exp(n\delta_G)$ .
- (2)  $\sharp(\Omega_r(go, n, \Delta) \asymp \exp(n\delta_G)$  for any  $g \in G$ .
- (3)  $\sharp(\Omega_{r, \epsilon, R}(go, n, \Delta) \asymp \exp(n\delta_G)$  for any  $g \in G$ .

*Remark.* In hyperbolic groups, the statements (1) and (2) were previously known in [4, Théorème 7.2] and in [1, Lemma 4] respectively. Without any assumption on the group  $G$ , we obtain analogous results for word metrics on a relatively hyperbolic group  $G$  in [21].

*Remark.* It is worth pointing out that, in [16, Théorème 4.1], a precise asymptotic formula of  $\sharp A(go, n, \Delta)$  was obtained instead of a bi-Lipschitz inequality in Corollary 1.8. However, Corollary 1.9 is sharp in a setting of coarse geometry: there exists a generating set  $S$  of  $G = PSL(2, \mathbb{Z})$  acting on  $\mathcal{G}(G, S)$  such that the limit of  $\frac{\sharp A(go, n, \Delta)}{\exp(\delta_G n)}$  does not exist. See [10, Section 3] for related discussions.

**1.3. Organization of paper.** Section 2 discusses some dynamical properties of a cusp-uniform action on boundaries and a notion of transition points relative to a contracting system. Theorem 1.4 is proved in Section 3, which is used to show the partial shadow lemma. Sections 4 and 5 prepare necessary ingredients in the proof of Theorem 1.7 in Section 6.

The large portion of this paper is essentially an extraction from a previous version of the paper [21], where Corollary 1.9 was proved under the assumption that  $G$  has the parabolic gap property. Some results in Section 3 found the analogous ones in [21]. We however include the detailed proof to address the differences and also make this paper independent.

*Acknowledgment.* The author is grateful to Marc Peigné for a helpful conversation in 2013 indicating him the results of T. Roblin, which leads to Theorem 1.7.

## 2. PRELIMINARIES

**2.1. Notations and Conventions.** Let  $(X, d)$  be a geodesic metric space. We collect some notations and conventions used globally in the paper.

- (1)  $B(x, r) := \{y : d(x, y) \leq r\}$  and  $N_r(A) := \{y \in X : d(y, A) \leq r\}$  for a subset  $A$  in  $X$ .
- (2)  $\|A\|$  denotes the diameter of  $A$  with respect to  $d$ .
- (3) All the paths we consider are rectifiable. Let  $p$  be a rectifiable path in  $X$  with arc-length parametrization. Denote by  $\ell(p)$  the length of  $p$ . Then  $p$  goes from the initial endpoint  $p_-$  to the terminal endpoint  $p_+$ . Let  $x, y \in p$  be two points which are given by parametrization. Then denote by  $[x, y]_p$  the parametrized subpath of  $p$  going from  $x$  to  $y$ .
- (4) Given a property (P), a point  $z$  on  $p$  is called the *first point* satisfying (P) if  $z$  is among the points  $w$  on  $p$  with the property (P) such that  $\ell([p_-, w]_p)$  is minimal. The *last point* satisfying (P) is defined in a similar way.
- (5) For  $x, y \in X$ , denote by  $[x, y]$  a geodesic  $p$  in  $X$  with  $p_- = x, p_+ = y$ . Note that the geodesic between two points is usually not unique. But the ambiguity of  $[x, y]$  is usually made clear or does not matter in the context.
- (6) Let  $p$  a path and  $Y$  be a closed subset in  $X$  such that  $p \cap Y \neq \emptyset$ . So the *entry* and *exit* points of  $p$  in  $Y$  are defined to be the first and last points  $z$  in  $p$  respectively such that  $z$  lies in  $Y$ .
- (7) Let  $f, g$  be two real-valued functions with domain understood in the context. Then  $f \prec_{c_1, c_2, \dots, c_n} g$  means that there is a constant  $C > 0$  depending on parameters  $c_i$  such that  $f < Cg$ . And  $f \succ_{c_1, c_2, \dots, c_n} g, f \asymp_{c_1, c_2, \dots, c_n} g$  are used in a similar way.

Recall that in a geodesic triangle, two points  $x, y$  in sides  $p, q$  respectively are called *congruent* if  $d(x, o) = d(y, o)$  where  $o$  is the common endpoint of  $p$  and  $q$ . Define the Gromov product  $(x, y)_z = (d(x, z) + d(y, z) - d(x, y))/2$  for  $x, y, z \in X$ .

We make use of the following definition of hyperbolic spaces in the sense of Gromov.

*Definition 2.1.* A geodesic space  $(X, d)$  is called  $\delta$ -hyperbolic for  $\delta \geq 0$  if any geodesic triangle is  $\delta$ -thin: let  $p, q$  be any two sides such that  $o := p_- = q_-$ . Then a point  $x$  in  $p$  such that  $d(x, p_-) \leq (p_+, q_+)_o$  is  $\delta$ -close to a congruent point in  $q$ .

**2.2. Dynamical formulation of cusp-uniform actions.** Sometimes, it is convenient to take a dynamical point of view to study the cusp-uniform action. We take [19], [2] as general references for these dynamical notions.

*Definition 2.2.* Let  $T$  be a compact metrizable space on which a group  $G$  acts by homeomorphisms.

- (1) The action of  $G$  on  $T$  is a *convergence group action* if the induced group action of  $G$  on the space of distinct triples over  $T$  is proper. The *limit set*  $\Lambda(\Gamma)$  of a subgroup  $\Gamma \subset G$  is the set of accumulation points of all  $\Gamma$ -orbits in  $T$ .
- (2) A point  $\xi \in T$  is called *conical* if there are a sequence of elements  $g_n \in G$  and a pair of distinct points  $a, b \in T$  such that the following holds

$$g_n(\xi, \zeta) \rightarrow (a, b),$$

for any  $\zeta \in T \setminus \xi$ . The set of all conical points is denoted by  $\Lambda^c G$ .

- (3) A point  $\xi \in T$  is called *bounded parabolic* if the stabilizer  $G_\xi$  in  $G$  of  $\xi$  is infinite, and acts properly and cocompactly on  $T \setminus \xi$ .
- (4) A convergence group action of  $G$  on  $T$  is called *geometrically finite* if every point  $\xi \in T$  is either a conical point or a bounded parabolic point.

Assume that  $G$  has a cusp-uniform action on a proper hyperbolic space  $(X, d)$ . It is well-known that  $G$  acts as a convergence group action on  $\partial X$ . Moreover, the cusp-uniform action is the same as the geometrically finite convergence action in the following sense.

*Theorem 2.3.* [3][20] *If  $G$  admits a cusp-uniform action on a proper hyperbolic space  $(X, d)$ , then  $G$  acts geometrically finitely on  $\Lambda G \subset \partial X$ . Conversely, if  $G$  acts geometrically finitely on  $T$  then  $G$  admits a cusp-uniform action on a proper hyperbolic space  $(X, d)$  such that  $T = \partial X$ .*

*Remark.* The first statement was proved by Bowditch in [3]; the second one was shown by Yaman in [20].

It is a useful fact that the stabilizer  $G_U$  of  $U \in \mathbb{U}$  acts cocompactly on the (topological) boundary  $\partial U$  in  $X$ . Note that the boundary at infinity of a horoball  $U$  in  $X \cup \partial X$  consists of a bounded parabolic point fixed by  $G_U$ . This gives the following observation.

*Observation 2.4.* Let  $\xi$  be a conical point in  $\partial X$ . Then for any  $\epsilon > 0$ , any geodesic ending at  $\xi$  exits the  $\epsilon$ -neighborhood of any horoball  $U \in \mathbb{U}$  which the geodesic enters into.

The following lemma is clear by the definition of cusp-uniform actions and the above observation.

*Lemma 2.5 (Conical points).* *There exists a constant  $r > 0$  with the following property.*

*A point  $\xi \in \Lambda G$  is a conical point if and only if there exists a sequence of elements  $g_n \in G$  such that for any geodesic ray  $\gamma$  in  $X$  with  $\gamma_+ = \xi$  and  $\gamma_- = o \in X$ , we have*

$$\gamma \cap B(g_n o, r) \neq \emptyset$$

*for all but finitely many  $g_n$ .*

**2.3. Contracting systems and transition points.** In this subsection, we turn to a notion of a contracting subset  $Y$  in a metric space  $X$  - a generalization of a quasi-convex subset in a hyperbolic space. The results obtained here apply to a hyperbolic space  $X$  with a collection of horoballs  $Y \in \mathbb{U}$ .

Given a subset  $Y$  in a metric space  $X$ , the projection  $\text{Pr}_Y(x)$  of a point  $x$  to  $Y$  is the set of nearest points in the closure of  $Y$  to  $x$ . Then for  $A \subset X$  define  $\text{Pr}_Y(A) = \cup_{a \in A} \text{Pr}_Y(a)$ .

*Definition 2.6.* Let  $\tau, D > 0$ . A subset  $Y$  is called  $(\tau, D)$ -contracting in  $X$  if the following holds

$$\|\text{Pr}_Y(\gamma)\| < D$$

for any geodesic  $\gamma$  in  $X$  with  $N_\tau(Y) \cap \gamma = \emptyset$ .

A collection of  $(\tau, D)$ -contracting subsets is referred to as a  $(\tau, D)$ -contracting system. The constants  $\tau, D$  will be often omitted, if understood.

*Remark.* If  $X$  is hyperbolic then the contracting property of a subset is equivalent to the quasi-convexity.

The following result is standard, with the proof left to the interested reader.

*Lemma 2.7.* For any  $\tau, D > 0$ , there exists a constant  $M = M(\epsilon, D) > 0$  with the following property.

Given a  $(\tau, D)$ -contracting subset  $Y$ , consider a geodesic  $\gamma$  with  $\gamma_+ \in Y$ , and denote by  $y$  the entry point of  $\gamma$  in  $Y$ . Then  $d(y, \text{Pr}_Y(\gamma_-)) \leq M$ .

We are interested in a contracting system  $\mathbb{Y}$  with a bounded intersection property: for any  $\epsilon > 0$  there exists  $R = R(\epsilon) > 0$  such that

$$\|N_\epsilon(Y) \cap N_\epsilon(Y')\| < R$$

for any two distinct  $Y, Y' \in \mathbb{Y}$ . By [22, Lemma 2.7], the bounded intersection property is equivalent to a bounded projection property: there exists a finite number  $D > 0$  such that

$$\|\text{Pr}_Y(Y')\| < D$$

for any two distinct  $Y, Y' \in \mathbb{Y}$ .

In the following definition, we state an abstract formulation of a notion of transition points, which was introduced originally by Hruska [11] in Cayley graphs.

*Definition 2.8.* Let  $\mathbb{Y}$  be a contracting system with the bounded intersection property in  $X$ . Fix  $\epsilon, R > 0$ . Given a path  $\gamma$ , we say that a point  $v$  in  $\gamma$  is called  $(\epsilon, R)$ -deep in  $Y \in \mathbb{Y}$  if it holds that  $\gamma \cap B(v, R) \subset N_\epsilon(Y)$ . If  $v$  is not  $(\epsilon, R)$ -deep in any  $Y \in \mathbb{Y}$ , then  $v$  is called an  $(\epsilon, R)$ -transition point in  $\gamma$ .

*Example 2.9.* In the definition of a cusp-uniform action, the system  $\mathbb{U}$  of horoballs is a contracting system with the bounded intersection property. More generally, any  $G$ -finite collection of uniformly quasi-convex subsets in  $X$  is a contracting system with the bounded intersection property.

We recall the following general facts about transition/deep points.

*Lemma 2.10.* [21, Lemma 2.8] There exists  $\epsilon_0 > 0$  such that for any  $R > 0$ , there exists  $L = L(R) > 0$  with the following property.

Let  $\gamma$  be a geodesic in  $X$ , and  $z \in \gamma$  such that  $d(z, \gamma_-) - d(\gamma_-, Y) > L$ ,  $d(z, \gamma_+) - d(\gamma_+, Y) > L$  for some  $Y \in \mathbb{Y}$ . Then  $z$  is  $(\epsilon_0, R)$ -deep in  $Y$ .

*Lemma 2.11.* [21, Lemma 2.9] For any  $\epsilon > 0$ , there exists  $R = R(\epsilon) > 0$  with the following property.

For a geodesic  $\gamma$  and  $Y \in \mathbb{Y}$ , the corresponding entry and exit points  $x, y$  of  $\gamma$  in  $N_\epsilon(Y)$  are  $(\epsilon, R)$ -transition points in  $\gamma$ , provided that  $d(x, y) \geq R$ .

From now on, we assume that  $X$  is a  $\delta$ -hyperbolic space and  $\mathbb{Y}$  is a contracting system with the bounded intersection property. We make ourselves conform to the following convention.



*Convention 2.12* (about  $\epsilon, R$ ). When talking about  $(\epsilon, R)$ -transition points we always assume that  $\epsilon \geq \max\{\epsilon_0, \delta\}$ , where  $\epsilon_0$  is given by Lemma 2.10. In addition, assume that  $R > R(\epsilon)$ , where  $R(\epsilon)$  is given by Lemma 2.11.

The main result here is the following analogue of Lemma 2.18 in [21].

*Lemma 2.13.* Let  $\epsilon, R > 0$  be in Convention (2.12). For any  $r > 0$ , there exist  $D = D(\epsilon, R), L = L(\epsilon, R, r) > 0$  with the following property.

Let  $\alpha, \gamma$  be two geodesics in a  $\delta$ -hyperbolic space  $X$  such that  $\alpha_- = \gamma_-$ ,  $d(\alpha_+, \gamma_+) < r$ . Take an  $(\epsilon, R)$ -transition point  $v$  in  $\alpha$  such that  $d(v, \alpha_+) > L$ . Then there exists an  $(\epsilon, R)$ -transition point  $w$  in  $\gamma$  such that  $d(v, w) < D$ .

*Proof.* Let  $w \in \gamma$  be the congruent point of  $v \in \alpha$ . Set  $L = L_0 + \epsilon + 2r$ , where  $L_0 = L(R)$  is given by Lemma 2.10. Since  $d(v, \alpha_+) > L > r$ , it follows that  $d(v, w) < \delta$  by the  $\delta$ -thin triangle property. Assume that  $w$  is  $(\epsilon, R)$ -deep in some  $Y \in \mathbb{Y}$ . Otherwise,  $w$  is an  $(\epsilon, R)$ -transition point and it is done.

Let  $x, y$  be the entry and exit points of  $\gamma$  in  $N_\epsilon(Y)$  respectively. Then  $x, y$  are  $(\epsilon, R)$ -transition points by Lemma 2.11.

*Claim.*  $\min\{d(w, x), d(w, y)\} \leq D := L_0 + 3\delta + \epsilon$ .

*Proof of the Claim.* Assume by contradiction that  $d(w, x), d(w, y) > L_0 + 3\delta + \epsilon$ . Since  $d(v, w) \leq \delta$ , we have  $d(v, x), d(v, y) > L_0 + 2\delta + \epsilon$ . Let  $x' \in \alpha$  be the congruent point of  $x \in \gamma$  so that  $d(x, x') \leq \delta$ . Then  $d(x', Y) \leq d(x, x') + d(x, Y) \leq \delta + \epsilon$  and  $d(v, x') \geq L_0 + \delta + \epsilon$ . Thus,  $d(v, x') - d(x', Y) \geq L_0$ .

Here are two cases to consider.

**Case (1):**  $d(y, \gamma_+) \leq r$ . Then  $d(\alpha_+, Y) \leq d(\alpha_+, \gamma_+) + d(y, \gamma_+) + d(y, Y) \leq 2r + \epsilon$ . So  $d(v, \alpha_+) - d(\alpha_+, Y) \geq L - 2r - \epsilon \geq L_0$ .

**Case (2):**  $d(y, \gamma_+) > r$ . By the  $\delta$ -thin triangle property, let  $y' \in \alpha$  be the congruent point of  $y \in \gamma$  so that  $d(y, y') \leq \delta$ . Then  $d(y', Y) \leq d(y, y') + d(y, Y) \leq \delta + \epsilon$  and  $d(v, y') \geq L_0 + \delta + \epsilon$ . Thus,  $d(v, y') - d(y', Y) \geq L_0$ .

In both cases, by Lemma 2.10, we see that  $v$  is  $(\epsilon, R)$ -deep in  $Y$ . This is a contradiction. The claim is proved.  $\square$

Since  $x, y$  are  $(\epsilon, R)$ -transition points in  $\gamma$ , the lemma follows from the claim.  $\square$

### 3. QUASI-CONFORMAL DENSITIES

In this section, assume that  $G$  has a cusp-uniform action on  $X$  such that  $G$  is of divergent type (Results of Coornaert in Subsection 3.2 even hold without this assumption). We fix the basepoint  $o \in X$ .

*Convention 3.1.* Let  $\mathbb{U} \subset \mathbb{Y}$  be a  $G$ -finite contracting system with bounded intersection property such that, for each  $Y \in \mathbb{Y}$ , the stabilizer  $G_Y$  acts co-compactly on either  $Y$  or  $\partial Y$ . We consider below the transition points defined with respect to  $\mathbb{Y}$ .

Before moving on, let's describe a typical example of  $\mathbb{Y}$  in Convention 3.1. We could consider an "extended" relative hyperbolic structure of  $G$ , which is obtained by adjoining into  $\mathcal{P}$  a collection of subgroups  $\mathcal{E}$ . This can be done in the following way. Let  $h \in G$  be a hyperbolic element. Denote by  $E(h)$  the stabilizer in  $G$  of the fixed points of  $h$  in  $\partial X$ . Then  $\mathcal{E} = \{gE(h)g^{-1} : g \in G\}$  gives such an example. See [13] for more detail.

Let  $C(E)$  denote the convex hull in  $X$  of  $\Lambda(E)$  for each  $E \in \mathcal{E}$ . Denote  $\mathbb{E} = \{C(E) : E \in \mathcal{E}\}$ . Then  $\mathbb{Y} := \mathbb{U} \cup \mathbb{E}$  is a contracting system with bounded intersection. Observe that a transition point relative to  $\mathbb{Y}$  is a transition point relative to  $\mathbb{U}$ .

**3.1. Partial shadows and cones.** These notions were introduced in [23].

*Definition 3.2* (Shadow and Partial Shadow). Let  $r, \epsilon, R \geq 0$  and  $g \in G$ . The *shadow*  $\Pi_r(go)$  at  $go$  is the set of points  $\xi \in \Lambda G$  such that there exists SOME geodesic  $[o, \xi]$  intersecting  $B(go, r)$ .

The *partial shadow*  $\Pi_{r, \epsilon, R}(go)$  is the set of points  $\xi \in \Pi_r(go)$  where, in addition, the geodesic  $[o, \xi]$  as above contains an  $(\epsilon, R)$ -transition point  $v$  in  $B(go, 2R)$ .

Inside the space  $X$ , the (partial) shadowed region motivates the notion of a (partial) cone.

*Definition 3.3* (Cone and Partial Cone). Let  $g \in G$  and  $r \geq 0$ . The *cone*  $\Omega_r(go)$  at  $go$  is the set of elements  $h$  in  $G$  such that there exists SOME geodesic  $\gamma = [o, ho]$  in  $X$  such that  $\gamma \cap B(go, r) \neq \emptyset$ .

The *partial cone*  $\Omega_{r, \epsilon, R}(go)$  at  $go$  is the set of elements  $h \in \Omega_r(go)$  such that one of the following statements holds.

- (1)  $d(o, ho) \leq d(o, go) + 2R$ ,
- (2) The geodesic  $\gamma$  as above contains an  $(\epsilon, R)$ -transition point  $v$  such that  $d(v, go) \leq 2R$ .

**3.2. Patterson-Sullivan measures.** The aim of this subsection is to recall some basic results of Patterson-Sullivan measures established by Coornaert in [4] for general discrete group actions on hyperbolic spaces.

A Borel measure  $\mu$  on a topological space  $T$  is *regular* if  $\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ is open}\}$  for any Borel set  $A$  in  $T$ . The  $\mu$  is called *tight* if  $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$  for any Borel set  $A$  in  $T$ .

Recall that *Radon* measures on a topological space  $T$  are finite, regular, tight and Borel measures. It is well-known that all finite Borel measures on compact metric spaces are Radon. Denote by  $\mathcal{M}(\Lambda G)$  the set of finite positive Radon measures on  $\Lambda G$ . Then  $G$  possesses an action on  $\mathcal{M}(\Lambda G)$  given by  $g_*\mu(A) = \mu(g^{-1}A)$  for any Borel set  $A$  in  $\Lambda G$ .

Endow  $\mathcal{M}(\Lambda G)$  with the weak-convergence topology. Write  $\mu(f) = \int f d\mu$  for a continuous function  $f \in C^1(\Lambda G)$ . Then  $\mu_n \rightarrow \mu$  for  $\mu_n \in \mathcal{M}(\Lambda G)$  if and only if  $\mu_n(f) \rightarrow \mu(f)$  for any  $f \in C^1(\Lambda G)$ , equivalently, if and only if,  $\liminf_{n \rightarrow \infty} \mu_n(U) \geq \mu(U)$  for any open set  $U \subset \Lambda G$ .

*Definition 3.4.* Let  $\sigma \in [0, \infty[$ . A  $G$ -equivariant map

$$\mu : G \rightarrow \mathcal{M}(\Lambda G), \quad g \rightarrow \mu_g$$

is a  $\sigma$ -dimensional *quasi-conformal density* if for any  $g, h \in G$  the following holds

$$(8) \quad \frac{d\mu_g}{d\mu_h}(\xi) \asymp \exp(-\sigma B_\xi(go, ho)),$$

for  $\mu_h$ -a.e. points  $\xi \in \Lambda G$ .

Here  $\mu$  is called  *$G$ -equivariant* if  $\mu_{hg}(A) = h_*\mu_g(A)$  for any Borel set  $A \subset \Lambda G$ .

*Remark.* Denote  $\nu = \mu_1$ , where  $1 \in G$  is the group identity. Define  $g^*\nu(A) = \nu(gA)$ . By the equivariant property of  $\mu$ , we obtain the following

$$(9) \quad \frac{dg^*\nu}{d\nu}(\xi) \asymp \exp(-\sigma B_\xi(g^{-1}o, o)),$$

for  $\nu$ -a.e. points  $\xi \in \Lambda G$ . The inequality (9) instead of (8) is used by some authors to define the quasi-conformal density, for example, in [4]. As  $G$  acts minimally on

$\Lambda G$  with  $\sharp \Lambda G > 3$ ,  $G$  has no global fixed point on  $\Lambda G$ . Then  $\mu_g$  is not an atom measure.

By the equivariant property of  $\mu$ , we see the following result.

*Lemma 3.5.* [4, Corollaire 5.2] *Let  $\{\mu_v\}_{v \in G}$  be a  $\sigma$ -dimensional quasi-conformal density on  $\Lambda G$ . Then the support of any  $\mu_v$  is  $\Lambda G$ .*

Here is a well-known construction, due to Patterson [15], of a quasi-conformal density. We start by constructing a family of measures  $\{\mu_v\}_{v \in G}^s$  supported on  $Go$  for any  $s > \delta_G$ . If  $\mathcal{P}_G(s, o)$  is divergent at  $s = \delta_G$ , set

$$\mu_v^s = \frac{1}{\mathcal{P}_G(s, o)} \sum_{g \in G} \exp(-sd(vo, go)) \cdot \text{Dirac}(go),$$

where  $s > \delta_G$  and  $v \in G$ . Note that  $\mu_1^s$  is a probability measure. Choose  $s_i \rightarrow \delta_G$  such that  $\mu_v^{s_i}$  are convergent in  $\mathcal{M}(\Lambda G)$ . Let  $\mu_v = \lim \mu_v^{s_i}$  be the limit measures, which are so called *Patterson-Sullivan measures*.

In the case that  $\mathcal{P}_G(s, o)$  is convergent at  $s = \delta_G$ , Patterson invented a trick to construct  $\mu_v^s$  similarly such that the limit  $\mu_v$  are supported on  $\Lambda G$ . Since in our case  $G$  is assumed to be divergent type, we omit the discussion of this case and refer the reader to [15].

In the sequel, we write PS-measures as shorthand for Patterson-Sullivan measures.

*Proposition 3.6.* [4, Théorème 5.4] *Let  $\{\mu_v\}_{v \in G}$  be a PS-measure on  $\Lambda G$  constructed through the action of  $G$  on  $X$ . Then  $\{\mu_v\}_{v \in G}$  is a  $\delta_G$ -dimensional quasi-conformal density.*

*Lemma 3.7 (Shadow Lemma).* [4, Proposition 6.1] *Let  $\{\mu_v\}_{v \in G}$  be  $\sigma$ -dimensional quasi-conformal density on  $\Lambda G$  for  $\sigma > 0$ . There exists  $r_0 > 0$  such that the following holds,*

$$\exp(-\sigma d(go, o)) < \mu_1(\Pi_r(go)) <_r \exp(-\sigma d(go, o)),$$

for any  $g \in G$  and  $r > r_0$ .

*Lemma 3.8 (Upper exponential growth).* [4, Proposition 6.4] *There exists  $\Delta_0 > 0$  such that the following holds*

$$\sharp A(n, \Delta) <_\Delta \exp(n\delta_G),$$

for any  $n \geq 0$  and  $\Delta > \Delta_0$ .

**3.3. No atoms at parabolic points.** Let  $\{\mu_v\}_{v \in G}$  be PS-measures on  $\Lambda G$ . We shall show that  $\mu_v$  has no atoms at parabolic points, following an argument of Dal'bo-Otal-Peigné in [5, Propositions 1 & 2].

Recall that the limit set  $\Lambda H$  of a subgroup  $H$  is the set of accumulation points of  $H$ . See Section 2 for more details.

The following lemma is proved by the same proof of the one in [21, Lemma 4.9]. We include the same proof for sake of completeness.

*Lemma 3.9.* *Let  $H$  be a subgroup in  $G$  such that  $\Lambda H$  is properly contained in  $\Lambda G$ . Then for any  $o \in X$ ,  $\mathcal{P}_H(s, o)$  is convergent at  $s = \delta_G$ . In particular, if  $H$  is divergent, then  $\delta_H < \delta_G$ .*

*Proof.* Since  $H$  acts properly on  $\Lambda G \setminus \Lambda H$ , choose a Borel fundamental domain  $K$  for this action such that the following holds

$$\mu_1(\Lambda G) = \sum_{h \in H} \mu_1(hK) + \mu_1(\Lambda H).$$

Observe that  $\mu_1(K) > 0$ . Indeed, if not, we see that  $\mu_1$  is supported on  $\Lambda H$ . This gives a contradiction, as  $\mu_1$  is supported on  $\Lambda G$ . Thus,  $\mu_1(K) > 0$ .

Note that  $\{\mu_v\}_{v \in G}$  is a  $\delta_G$ -dimensional quasi-conformal density. It follows that

$$\mu_1(h^{-1}K) = \mu_h(K) > \exp(-\delta_G d(o, ho)) \cdot \mu_1(K)$$

for any  $h \in H$ . Hence,

$$\mu_1(\Lambda G) > \sum_{h \in H} \mu_1(hK) > \sum_{h \in H} \exp(-\delta_G d(o, ho)) \cdot \mu_1(K),$$

implying that  $\sum_{h \in H} \exp(-\delta_G d(o, ho))$  is finite. This concludes the proof.  $\square$

Recall that  $\partial X$  denotes the Gromov boundary of  $X$ . Given a discrete subset  $U \subset X$ , denote by  $\Lambda U$  the boundary of  $U$  at infinity in  $\partial X$ .

*Lemma 3.10.* *Fix a basepoint  $o \in X$ . There exists a constant  $M = M(Go) > 0$  with the following property. Let  $U, V \subset Go$  such that  $\Lambda U \cap \Lambda V = \emptyset$ . Then  $\|Pr_V(U)\| \leq M$ .*

*Proof.* Fix a point  $v \in V$ . For any  $u \in U$ , denote by  $\bar{u} \in V$  a projection point of  $u$  to  $V$ . We claim that there exists a finite number  $M > 0$  such that  $d(v, [u, \bar{u}]) \leq M$ . Argue by way of contradiction. Assume that there exist infinitely many pairs  $(u_n, \bar{u}_n) \in U \times V$  such that  $d(v, [u_n, \bar{u}_n]) \geq n$  for  $n \in \mathbb{N}$ . Without loss of generality, assume that  $u_n, \bar{u}_n$  converge to  $\xi, \eta \in \partial X$  respectively. Since  $d(v, [u_n, \bar{u}_n]) \geq n$ , we see that  $\xi = \eta$  by definition of visual metric. However, this is a contradiction, as  $\Lambda U \cap \Lambda V = \emptyset$ . Thus, the claim is proved.

Recall that  $\bar{u}$  is a shortest point in  $V$  from  $u$ . By the claim, it follows easily that  $d(v, \bar{u}) \leq 2M$  for any  $u \in U$ . Since  $Go$  is a discrete set, the conclusion follows.  $\square$

By Lemma 3.10, the following result is proved with the almost same proof as [21, Lemma 4.10].

*Lemma 3.11.* *Assume that  $G$  is of divergent type. Then  $\{\mu_v\}_{v \in G}$  have no atoms at bounded parabolic points.*

*Proof.* Let  $q \in \partial X$  be a bounded parabolic point, with the stabilizer  $P$  in  $G$  and an open  $U \in \mathbb{U}$  such that  $\partial U = \{q\}$ . Choose a compact set  $K$  in  $X \cup \partial X$  such that  $P \cdot K = (X \setminus U) \cup \partial X \setminus \{q\}$ . Indeed, since  $P$  acts co-compactly on  $\partial X \setminus \{q\}$  with a compact fundamental domain  $Q \subset \partial X \setminus \{q\}$ . Consider the set  $H(Q)$  in  $X$  which is the union of all geodesics with one endpoint in  $Q$  and the other one at  $q$ . Then  $K := (H(Q) \cap X) \cup Q$  is compact and  $P \cdot K = (X \setminus U) \cup \partial X \setminus \{q\}$  by construction.

We may choose the basepoint  $o \in \partial U$  for convenience. We can further assume that the boundary of  $K$  is  $\mu_v$ -null for some (hence any)  $v \in G$ . Let

$$V_n = \bigcup_{d(o, po) > n, p \in P} pK.$$

Then  $V_n \cup \{q\}$  is a decreasing sequence of open neighborhoods of  $q$ . Note that the boundary of  $V_n$  is  $\mu_1$ -null. It follows that  $\mu_1^s(V_n) \rightarrow \mu_1(V_n)$  for each  $V_n$ , as  $s \rightarrow \delta_G$ .

By Lemma 2.7, there exists  $D > 0$  such that

$$(10) \quad d(Pr_U(z), [x, z]) \leq D$$

for any  $x \in \partial U, z \in X$ . Since  $\|\text{Pr}_{Po}(K \cap Go)\| < \infty$  by Lemma 3.10, and  $P$  acts co-compactly on  $\partial U$ , we can assume that  $\|\text{Pr}_U(K \cap Go)\| < D$  for the same constant  $D$ . Up to a translation by an element in  $P$ , we further assume that  $o \in \text{Pr}_U(K \cap Go)$ .

Note that  $\text{Pr}_U(z) \subset \text{Pr}_U(K \cap Go)$  for any  $z \in K \cap Go$ . By using (10) for  $x = po$ , a twice repetition of the triangle inequality gives

$$(11) \quad d(z, po) + 2D > d(z, o) + d(o, po) > d(po, z),$$

for any  $p \in P$ . Since  $G$  is of divergent type, we estimate the measure  $\mu_1^s$  by (11):

$$\begin{aligned} \mu_1^s(V_n) &= \mu_1^s \left( \bigcup_{d(o, po) > n}^{p \in P} pK \right) \leq \sum_{d(o, po) > n}^{p \in P} \mu_1^s(pK) \\ &\leq \frac{1}{\mathcal{P}_G(s, o)} \sum_{d(o, po) > n}^{p \in P} \left( \sum_{g \in G, go \in pK} \exp(-sd(o, go)) \cdot \text{Dirac}(go) \right) \\ &\leq \frac{\exp(2sD)}{\mathcal{P}_G(s, o)} \sum_{d(o, po) > n}^{p \in P} \left( \sum_{g \in G, go \in pK} \exp(-sd(po, go) - sd(o, po)) \cdot \text{Dirac}(go) \right) \end{aligned}$$

yielding

$$\begin{aligned} \mu_1^s(V_n) &\leq \frac{\exp(2sD)}{\mathcal{P}_G(s, o)} \sum_{d(o, po) > n}^{p \in P} \exp(-sd(o, po)) \sum_{g \in G, go \in pK} \exp(-sd(o, go)) \cdot \text{Dirac}(go) \\ &\leq \exp(2sD) \cdot \sum_{d(o, po) > n}^{p \in P} \exp(-sd(o, po)) \cdot \mu_1^s(K). \end{aligned}$$

Letting  $s \rightarrow \delta_G$  we have  $\mu_1^s(V_n) \rightarrow \mu_1(V_n)$ . By Lemma 3.9,  $\mathcal{P}_P(s, o)$  is convergent at  $s = \delta_G$  so that  $\mu_1(V_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence,  $\mu_1(q) = 0$ .  $\square$

**3.4. No atoms at conical points.** Fix  $\Delta > 1$  and consider  $G = \cup_{i \geq 1} A(o, i, \Delta)$ , where  $A(o, i, \Delta)$  is given in (3). By Lemma 2.5, we have that  $\Lambda^c G = A_r$  for any fixed  $r \gg 0$ , where

$$(12) \quad A_r := \bigcap_{n=1}^{\infty} \bigcup_{g \in \cup_{i \geq n} A(o, i, \Delta)} \Pi_r(go).$$

In other words, a conical point is shadowed infinitely many times by the orbit  $Go$ .

*Lemma 3.12. Conical points are not atoms of PS-measures.*

*Proof.* Note that a conical point  $\xi$  lies in infinitely many shadows  $\Pi_r(go)$  for  $g \in G$ . As  $\mu_1(\Pi_r(go)) \rightarrow 0$  as  $d(o, go) \rightarrow \infty$  we have that  $\mu_1(\xi) = 0$ .  $\square$

*Lemma 3.13. Let  $\{\mu_v\}_{v \in G}$  be a  $\sigma$ -dimensional quasi-conformal density on  $\Lambda G$ . If for some (thus any  $v \in G$ )  $\mu_v$  gives positive measure to the set of conical points, then  $\mathcal{P}_G(s, o)$  is divergent at  $s = \delta_G$ . In particular  $\sigma \leq \delta_G$ .*

*Proof.* We fix  $r > r_0$  so that  $\Lambda^c G \subset A_r$ , where  $r_0$  is given by Lemma 3.7. Then  $\mu_1(A_r) \geq \mu_1(\Lambda^c G) > 0$ . By an argument as in Lemma 5.5 we have

$$\mathcal{P}_G(s, o) = \sum_{g \in G} \exp(-sd(o, go)) \asymp_{\Delta} \sum_{i=0}^{\infty} \# A(o, i, \Delta) \exp(-si).$$

We claim that  $\mathcal{P}_G(s, 1)$  is divergent at  $s = \sigma$ . Indeed, by Lemma 3.7 the following holds

$$\begin{aligned} \sum_{i=n}^{\infty} \# A(o, i, \Delta) \exp(-\sigma i) &> \sum_{i=n}^{\infty} \sum_{g \in A(o, i, \Delta)} \mu_1(\Pi_r(go)) \\ &> \mu_1(\cup_{g \in \cup_{i \geq n} A(o, i, \Delta)} \Pi_r(go)) \\ &> \mu_1(A_r) \end{aligned}$$

for any  $n > 0$ . Since  $\mu_1(A_r)$  is a fixed positive number,  $\mathcal{P}_G(s, 1)$  is divergent at  $s = \sigma$ . This completes the proof.  $\square$

*Corollary 3.14.* *If  $\{\mu_v\}_{v \in G}$  is a  $\sigma$ -dimensional quasi-conformal density for  $G$  giving positive measure on conical points, then  $\sigma = \delta_G$ .*

As  $G$  acts geometrically finitely on  $\Lambda G$ , there exist only bounded parabolic points and conical points in  $\Lambda G$ . Hence, Lemmas 3.11 and 3.12 together prove the following proposition, where the “moreover” statement is proved in [21, Appendix: Proposition A.4].

*Proposition 3.15.* *Assume that  $G$  is of divergent type. Then PS-measures  $\{\mu_v\}_{v \in G}$  are  $\delta_G$ -dimensional quasi-conformal density without atoms. Moreover,  $\mu$  is unique and ergodic.*

**3.5. Partial Shadow Lemma.** By Convention 3.1, let  $\mathbb{Y}$  be a  $G$ -finite contracting system with the bounded intersection property so that  $\mathbb{U} \subset \mathbb{Y}$ , for which we consider the transition points. We shall prove a variant of Shadow Lemma called *Partial Shadow Lemma* with respect to the so-defined transition points.

Let’s prepare some preliminary results. Recall that  $o_Y \in \partial Y$  is a projection point of  $o$  to  $Y \in \mathbb{Y}$ . The following lemma follows by definition of a cusp-uniform action.

*Lemma 3.16.* *There exists a constant  $M > 0$  with the following property.*

*For each  $Y \in \mathbb{Y}$ , there exists  $t_Y \in G$  such that  $d(t_Y o, o_Y) \leq M$  and  $\partial Y \subset N_M((G_Y t_Y) \cdot o)$ .*

Note that  $Y \in \mathbb{Y}$  is contracting and thus quasi-convex in a hyperbolic space  $X$ . Since  $\mathbb{Y}$  has the bounded intersection property, we see that the closure of  $Y$  in  $\partial X$  is a proper subset so that the limit set of  $G_Y$  is also proper. By Convention 3.1,  $G_Y$  acts co-compactly on either  $\partial Y$  or  $Y$ . So the following observation follows from Lemma 3.9.

*Lemma 3.17.* *For any  $\epsilon, r > 0$ , there exists  $R = R(\epsilon, r) > 0$  such that the following holds*

$$\sum_{h \in G_Y}^{d(z, hw) > R} \exp(-\delta_G d(z, hw)) < \epsilon,$$

*for any  $Y \in \mathbb{Y}$  and any  $z, w \in N_r(\partial Y)$ .*

We now state the partial shadow lemma. We remark that the proof relies crucially on the fact that  $\mu_1$  has no atoms at parabolic points proven in Lemma 3.11, which in turn is proved using the divergence of the action of  $G$  on  $X$ .

*Lemma 3.18 (Partial Shadow Lemma).* *Let  $\mathbb{Y}$  be a contracting system given in Convention 3.1. There are constants  $r, \epsilon, R \geq 0$  such that the following holds*

$$(13) \quad \exp(-\delta_G d(o, go)) < \mu_1(\Pi_{r, \epsilon, R}(go)) <_r \exp(-\delta_G d(o, go)),$$

*for any  $g \in G$ .*

*Remark.* In this paper we only apply the lemma to the case  $\mathbb{P} = \mathbb{U}$ . However, a choice of a larger contracting system  $\mathbb{Y}$  becomes essential in [21].

*Proof.* Let  $M$  given by Lemma 3.16. Given any  $g \in G$ , there exist  $r > M, C_1, C_2 > 0$  be given by Shadow Lemma 3.7 such that

$$(14) \quad C_1 \exp(-\delta_G d(o, go)) \leq \mu_1(\Pi_r(go)) \leq C_2 \exp(-\delta_G d(o, go)).$$

Denote by  $\mathbb{F}$  the set of  $Y \in \mathbb{Y}$  such that  $Y \cap B(go, r + \epsilon) \neq \emptyset$ . Since  $\mathbb{Y}$  is locally finite, we have that  $\#\mathbb{F}$  is a uniform number depending only on  $G$ . The choice of the constant  $R > 0$  will be made in the remainder of proof.

Denote  $\Xi := \Pi_r(go) \setminus \Pi_{r, \epsilon, R}(go)$ . For any  $\xi \in \Xi$ , any geodesic  $\gamma = [o, \xi]$  does not contain an  $(\epsilon, R)$ -transition point in the ball  $B(go, 2R)$ .

Choose  $x \in B(go, r) \cap \gamma$  such that  $d(go, x) < r$ . Thus,  $\gamma$  does not contain an  $(\epsilon, R)$ -transition point in  $B(o, 2R - r)$ . Since  $d(go, Y) < r + \epsilon$ , we have  $Y \in \mathbb{F}$ .

Note that a bounded parabolic point is not an atom by Lemma 3.11. Without loss of generality, assume that  $\xi$  is a conical point so that  $\gamma$  exits every  $Y \in \mathbb{Y}$ . Let  $z$  be the exit point of  $\gamma$  in  $N_\epsilon(Y)$  so that  $d(z, x) \geq R$ . Choose  $R > R_1$ , where  $R_1 > 0$  is given by Lemma 2.11. Hence,  $z$  is an  $(\epsilon, R)$ -transition point in  $[o, \xi]$ . Since  $d(go, x) < r$ , we see that  $d(z, x) > 2R - r$ ; otherwise,  $B(go, 2R)$  contain an  $(\epsilon, R)$ -transition point in  $\gamma$ .

By Lemma 3.16, there exists  $h \in G_Y$  and  $t_Y \in G$  such that  $d(h \cdot t_Y o, z) \leq M \leq r$ . This implies

$$(15) \quad \begin{aligned} d(h \cdot t_Y o, go) &> d(z, x) - d(z, h \cdot t_Y o) - d(x, go) \\ &\geq 2R - 3r. \end{aligned}$$

Note that  $t_Y o, go \in N_{r+\epsilon}(\partial Y)$ . By Lemma 3.17, there exists  $R_2 > 0$  depending on  $r, \epsilon, \#\mathbb{F}$  such that

$$(16) \quad \sum_{h \in G_Y}^{d(h \cdot t_Y o, go) > R_2} \exp(-\delta_G d(go, h \cdot t_Y o)) \cdot \#\mathbb{F} \cdot \exp(4\delta_G r) < C_1/(2C_2).$$

Since  $z \in [x, \xi]_\gamma$  and  $d(go, x) \leq r$ , we have  $d(o, z) + 2r \geq d(o, go) + d(go, z)$ . Hence,

$$d(o, h \cdot t_Y o) \geq d(o, z) - r \geq d(o, go) + d(go, h \cdot t_Y o) - 4r$$

We assume that  $2R - 3r \geq R_2$ . By (15) and (16), the following holds:

$$(17) \quad \begin{aligned} \mu_1(\Xi) &\leq \sum_{Y \in \mathbb{F}} \left( \sum_{h \in G_Y}^{d(h \cdot t_Y o, go) > R_2} \mu_1(\Pi_r(h \cdot t_Y o)) \right) \\ &\leq C_2 \cdot \sum_{Y \in \mathbb{F}} \left( \sum_{h \in G_Y}^{d(h \cdot t_Y o, go) > R_2} \exp(-\delta_G d(o, h \cdot t_Y o)) \right) \\ &\leq C_2 \cdot \exp(-\delta_G d(o, go)) \cdot \sum_{Y \in \mathbb{F}} \sum_{h \in G_Y}^{d(h \cdot t_Y o, go) > R_2} \exp(-\delta_G d(go, h \cdot t_Y o)) \cdot \exp(4\delta_G r) \\ &\leq \exp(-\delta_G d(1, g)) \cdot C_1/2. \end{aligned}$$

Notice that  $\mu_1(\Xi) + \mu_1(\Pi_{r, \epsilon, R}(go)) = \mu_1(\Pi_r(go)) \geq C_1 \exp(-\delta_G d(o, go))$ . So the inequalities (14) and (17) yield

$$\mu_1(\Pi_{r, \epsilon, R}(go)) \geq (C_1/2) \cdot \exp(-\delta_G d(o, go)).$$

The proof is now complete.  $\square$

#### 4. HOROBALL GROWTH FUNCTIONS

In this section, we prove the direction “(2) $\Leftrightarrow$ (4)” of Theorem 1.7.

From now on, we assume that  $\partial X = \Lambda G$  for simplicity so that  $G$  acts co-compactly on  $X \setminus \mathcal{U}$ . In what follows, we fix  $M > 0$  a constant simultaneously satisfying Lemmas 2.7, 3.16 such that

$$(18) \quad M \geq \|(X \setminus \mathcal{U})/G\|.$$

Let  $t_U \in G$  be given by Lemma 3.16 for  $U \in \mathbb{U}$  such that  $d(o_U, t_U o) \leq M$ . So if  $G$  has purely exponential horoball growth, then  $G$  has purely exponential orbit growth. In fact, the converse statement is also true.

*Lemma 4.1. Suppose that the orbit growth function is purely exponential. Then the horoball growth function of  $G$  is purely exponential. Moreover, the horoball growth function of  $G$  of type  $U \in \mathbb{U}$ :*

$$\sharp H_U(o, n, \Delta) := \sharp \{V \in \mathbb{U} : |d(o, V) - n| \leq \Delta; \exists g \in G : gU = V\}$$

*is purely exponential for some  $\Delta > 0$ .*

*Proof.* By assumption, there exist  $C, \Delta > 0$  such that

$$(19) \quad C \exp(\delta_G n) \geq \sharp A(o, n, \Delta + M) \geq C^{-1} \exp(\delta_G n)$$

for  $n \geq 1$ . Consider  $go \in A(o, n, \Delta)$ . Since  $\|X \setminus \mathcal{U}/G\| \leq M$ , there exist  $U \in \mathbb{U}$  and  $x \in \partial U$  such that  $d(go, x) \leq M$ . Let  $o_U$  be a projection point of  $o$  to  $U$ . Choose

$$R_0 > \max\{2M, \mathcal{R}(2(M + \delta)) + 2(M + \delta)\}.$$

*Claim.* If  $d(x, o_U) \geq R_0$ , then the choice of  $U$  as above is unique.

*Proof of the Claim.* Assume by contradiction that there exists a distinct  $U \neq V \in \mathbb{U}$  and  $y \in \partial V$  such that  $d(go, y) \leq M$ . By Lemma 2.7, there exists  $z \in [o, x]$  such that  $d(o_U, z) \leq M$ . Since  $d(x, go) \leq M$ , the following holds

$$d(z, [o, go]) \leq \delta$$

by the  $\delta$ -thin triangle property. We then have  $d(o_U, [o, go]) \leq M + \delta$ . Note that

$$[o, go] \cap N_{M+\delta}(U) \geq d(x, o_U) - 2(M + \delta) > \mathcal{R}(2(M + \delta)).$$

The same reasoning gives  $[o, go] \cap N_{M+\delta}(V) > \mathcal{R}(2(M + \delta))$ . This implies  $N_{M+\delta}(U) \cap N_{M+\delta}(V) \geq \mathcal{R}(2(M + \delta))$ . We got a contradiction.  $\square$

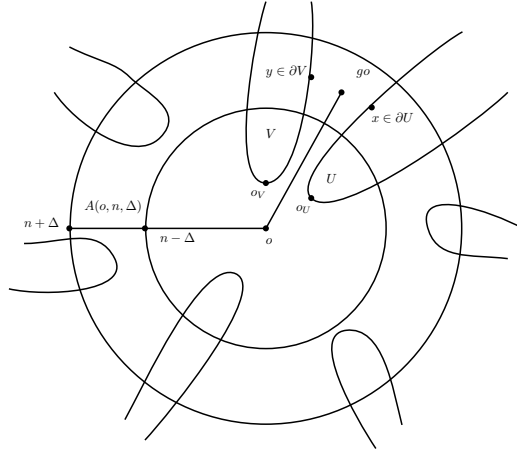


FIGURE 1. Lemma 4.1

By Lemma 3.17, there exists  $R > R_0$  such that

$$(20) \quad \sum_{n \geq R} \sharp A_U(o_U, n, \Delta) \cdot \exp(-\delta_G n) \leq \frac{1}{2C^2},$$



for any  $U \in \mathbb{U}$ . Denote by  $\mathbb{B}$  the set of  $go \in A(o, n, \Delta)$  such that

$$d(go, o_U) \geq R + 2\Delta,$$

where  $U$  is given for  $go$  as above and is unique by the Claim. We shall show that  $\#\mathbb{B} \leq \frac{\exp(n\delta_G)}{2C}$ . For this purpose, we decompose the set  $\mathbb{B}$  as the union of *layers*  $B_i(o, n, \Delta)$  for  $0 \leq i \leq n - R$ , which is the set of  $go \in \mathbb{B}$  such that:

$$|d(o, o_U) - i| \leq \Delta.$$

By Lemma 3.16 there exists  $t_U \in G$  such that  $d(o_U, t_U o) \leq M$ . A simple calculation shows

$$\#B_i(o, n, \Delta) \leq \#A(o, i, \Delta + M) \cdot \#A_U(t_U o, n - i, 2\Delta + M)$$

yielding:

$$\#B_i(o, n, \Delta) < C \cdot \frac{\#A_U(t_U o, n - i, \Delta + M)}{\exp((n - i)\delta_G)} \cdot \exp(n\delta_G).$$

Since  $go \in A(o, n, \Delta)$  we have  $d(o, go) < n + \Delta$ . Thus,  $n - i \geq n - d(o, o_U) - \Delta \geq d(o, go) - d(o, o_U) - 2\Delta > R$ . We sum up  $\#B_i(o, n, \Delta)$  over  $i$  and then use (20) to get

$$\#\mathbb{B} < C \exp(n\delta_G) \cdot \sum_{j \geq R} \frac{\#A_U(o_U, j, \Delta)}{\exp(j\delta_G)} \leq \frac{\exp(n\delta_G)}{2C}.$$

By (19),  $\#(A(o, n, \Delta + M) \setminus \mathbb{B}) \geq \frac{\exp(n\delta_G)}{2C}$ . Since for each  $go \in A(o, n, \Delta + M) \setminus \mathbb{B}$  we have  $d(go, U) \leq R + 2\Delta$  and whence there are only finitely many  $U \in \mathbb{U}$  sharing a common  $go$ . So we see that

$$\#H(o, n, R + 3\Delta) > \#(A(o, n, \Delta + M) \setminus \mathbb{B}),$$

proving the lower bound. The upper bound of  $\#H(o, n, R + 3\Delta)$  is clear, for instance by an argument in (3) of Lemma 5.4. So the purely exponential growth for all horoballs is proved.

For the “moreover” statement, we fix  $U \in \mathbb{U}$  and observe that for each point  $go \in A(o, n, \Delta)$ , there exists  $g' \in G$  such that  $d(g'U, go) \leq M$ . This is because  $Go$  is contained in the  $M$ -neighbourhood of the union of  $U \in \mathbb{U}$ . So the growth of horoballs in the same  $G$ -orbit is also purely exponential. The lemma is proved.  $\square$

## 5. GROWTH OF THE ORBIT IN PARTIAL CONES

This section prepares the necessary ingredients in the proof of Theorem 1.7 in Section 6. The idea of proof is using the shadows of orbit vertices in the annulus  $A(go, n, \Delta)$  to cover  $\Pi_{r, \epsilon, R}(go)$ . However, the existence of “solid” horoballs causes a particular difficulty: the annulus around  $go$  may not be uniformly spaced by the orbit  $Go$ . It is this place where we put effort into the analysis of distribution of horoballs in cones, and also where the DOP condition takes into action.

**5.1. Annular and horospherical shadows.** We introduce the main technical definition and notations in the proof of Theorem 1.7.

*Definition 5.1.* Let  $r \geq 0, g \in G$ . For  $n \geq 0$ , consider the collection of horoballs  $U \in \mathbb{U}$ , denoted by  $\mathbb{U}_{r, n}(go)$ , for which there exists a geodesic  $\gamma = [o, \xi]$  for  $\xi \in \Pi_r(go)$  such that  $U$  contains a point  $z \in \gamma$  having  $d(o, z) - d(o, go) = n$ .

Fix  $U \in \mathbb{U}_{r, n}(go)$  and  $L > 0$ . Let  $\xi$  be a conical point in  $\Pi_r(go)$  and  $\gamma = [o, \xi]$  be a geodesic such that the following hold

- (1) there exists  $U \in \mathbb{U}$  such that  $U$  contains a point  $z \in \gamma$  having  $d(o, z) - d(o, go) = n$ .
- (2) consider the exit point  $z_+$  of  $\gamma$  in  $U$  such that  $d(z, z_+) \geq L$ . By Lemma 3.16, there exist  $h \in G_U, t_U \in G$  such that  $d(h \cdot t_U o, z_+) < M$  (18).

Denote by  $G_{U,L}$  the set of all elements  $h$  satisfying (2).

Recall that  $\Pi_{r,\epsilon,R}^c(go)$  denotes the set of conical limit points in  $\Pi_{r,\epsilon,R}(go)$ . Since  $G$  is of divergent type, we have that  $\mu_1$  has no atoms at parabolic points by Lemma 3.11. So  $\mu_1(\Pi_{r,\epsilon,R}^c(go)) = \mu_1(\Pi_{r,\epsilon,R}(go))$ . The constant  $\delta$  below is the hyperbolicity constant of  $X$ .

*Lemma 5.2. There exist  $\epsilon, r_0 > 0$  such that the following holds.*

*For any  $R, L > 0$ , there exist  $R' = R(\epsilon, R), \Delta = \Delta(L) > 0$  such that the set  $\Pi_{r,\epsilon,R}^c(go)$  is covered by the union of the following two types of shadows:*

- (1) the **annular** shadows:  $\{\Pi_r(ho) : h \in \Omega_{r+\delta,\epsilon,R'}(go, n, \Delta)\},$
- (2) the **horospherical** shadows:  $\{\Pi_r(h \cdot t_U o) : h \in G_{U,L}, U \in \mathbb{U}_{r,n}(go)\}$

*for any  $n \gg 0, r > r_0$ . Similarly,  $\Pi_r(go)$  is covered by the union of*

- (1) the **annular** shadows:  $\{\Pi_r(ho) : h \in \Omega_{r+\delta}(go, n, \Delta)\},$
- (2) the **horospherical** shadows:  $\{\Pi_r(h \cdot t_U o) : h \in G_{U,L}, U \in \mathbb{U}_{r,n}(go)\}$

*which is contained in  $\Pi_{r+\delta,\epsilon,R}(go)$ .*

*Proof.* We fix  $L, R > 0$ . Set  $r_0 = M$  and  $\Delta = L + M$ . For any  $\xi \in \Pi_{r,\epsilon,R}^c(go)$ , there exists a geodesic  $\gamma = [o, \xi]$  such that  $\gamma$  contains an  $(\epsilon, R)$ -transition point  $v$  in  $B(go, 2R)$ . Take  $z \in \gamma$  such that

$$(21) \quad d(o, z) - d(o, go) = n.$$

We have two cases to consider as follows.

**Case 1.** There exists  $h \in G$  such that  $d(ho, z) \leq M$  for  $M > 0$  in (18). It follows that

$$(22) \quad \xi \in \Pi_M(ho) ; \quad \Pi_M(ho) \subset \Pi_r(ho).$$

Let  $L_1 = L(\epsilon, R, M), D = D(\epsilon, R)$  be given by Lemma 2.13. Choose

$$n > \max\{L_1 + 2R, M\}.$$

Since  $d(z, ho) \leq M$  and  $n > M$ , we then use the  $\delta$ -thin triangle property to get

$$d(go, [o, ho]) < r + \delta.$$

By (21), we obtain  $d(z, v) \geq d(z, go) - d(go, v) > n - 2R > L_1$ . By Lemma 2.13, there exists an  $(\epsilon, R)$ -transition point  $w$  in  $[o, ho]$  such that  $d(v, w) < D$ . Then  $d(w, go) < 2R + D$ , which implies

$$(23) \quad h \in \Omega_{r+\delta,\epsilon,2R+D}(go, n, \Delta).$$

Denote  $R' = 2R + D$ . Hence by (22) and (23), we see that  $\xi$  lies in a shadow of annular type.

**Case 2.** The ball  $B(z, M)$  contains no point in  $Go$ . Since  $M \geq \|(X \setminus \mathcal{U})/G\|$ , there exists  $U \in \mathbb{U}_{r,n}(go)$  such that  $z \in U$ .

As  $\xi$  is a conical point, any geodesic  $[o, \xi]$  leaves every horoball into which it enters. Let  $z_+ \in \partial U$  be the exit point of  $[o, \xi]$  in  $U$ . By Lemma 3.16, there exist  $h \in G_U, t_U \in G$  such that  $d(ht_U o, z_+) < M$ . So

$$(24) \quad \xi \in \Pi_r(h \cdot t_U o).$$

Since  $n > M$ , by the  $\delta$ -thin triangle property we have  $d(go, [o, ht_U o]) < r + \delta$  and so  $ht_U \in \Omega_{r+\delta}(go)$ .

To finish the proof, we continue to examine two subcases of **Case 2** as follows.

*Subcase 1:* assume that  $d(z, z_+) \leq L$ . Then  $d(z, ht_U o) \leq M + L \leq \Delta$ . Note that  $d(go, [o, ht_U o]) < r + \delta$ . Arguing similarly as in the **Case 1**, we see that there exists an  $(\epsilon, R)$ -transition point  $w$  in  $[o, h \cdot t_U o]$  such that  $d(go, w) < 2R + D$ . This implies

$$(25) \quad ht_U \in \Omega_{r+\delta, \epsilon, R'}(go, n, \Delta).$$

Hence by (24) and (25),  $\xi$  also goes into a shadow of annular type.

*Subcase 2:* assume that  $d(z, z_+) > L$ . By definition,  $h \in G_{U, L}$ . By (24), this shows that  $\xi$  is in a shadow of horospherical type. See Figure 2.

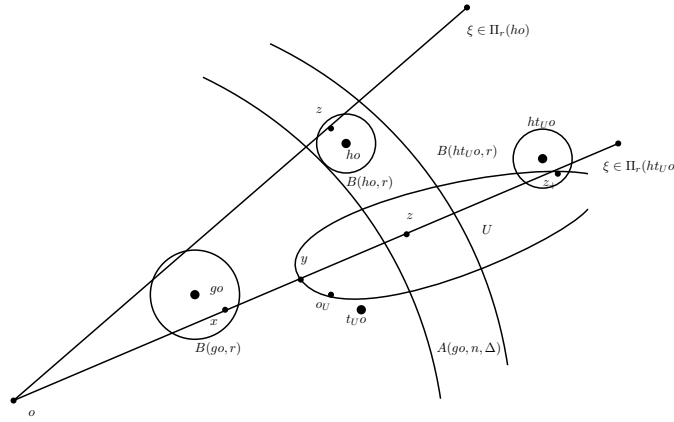


FIGURE 2. Lemma 5.2

The same argument applies to  $\Pi_r(go)$  and the statement that the union of (1) and (2) is contained in  $\Pi_{r+\delta}(go)$  follows by the  $\delta$ -thin triangle property. The lemma is proved.  $\square$

By noting that  $\mu_1(\Pi_{r, \epsilon, R}^c(go)) = \mu_1(\Pi_{r, \epsilon, R}(go))$ , we deduce the following result from Lemma 5.2.

**Lemma 5.3.** *There exist  $\epsilon, r_0 > 0$  with the following property.*

*For any  $R, L > 0$ , there exist  $R' = R(\epsilon, R), \Delta = \Delta(L) > 0$  such that the following holds*

$$(26) \quad \mu_1(\Pi_{r, \epsilon, R}(go)) <_r \left( \sum_{h \in \Omega_{r+\delta, \epsilon, R'}(go, n, \Delta)} \mu_1(\Pi_r(ho)) \right) + \left( \sum_{h \in G_{U, L}}^{U \in \mathbb{U}_{r, n}(go)} \mu_1(\Pi_r(h \cdot t_U o)) \right),$$

and

$$(27) \quad \mu_1(\Pi_r(go)) \asymp_r \left( \sum_{h \in \Omega_{r+\delta}(go, n, \Delta)} \mu_1(\Pi_r(ho)) \right) + \left( \sum_{h \in G_{U, L}}^{U \in \mathbb{U}_{r, n}(go)} \mu_1(\Pi_r(h \cdot t_U o)) \right),$$

for any  $r \geq r_0$  and  $n \geq 0$ .

*Proof.* The direction “ $<_r$ ” follows from the first statement in Lemma 5.2 (in fact we have the inequality “ $\leq$ ”).

Recall the following fact in [4, Lemme 6.5]:

*Claim.* For any  $r > 0$ , there exists  $N = N(r)$  such that for any  $n \geq 0$  a point  $\xi \in \partial X$  is covered at most  $N$  times by shadows from the collection  $\{\Pi_r(go) : g \in A(o, n, \Delta)\}$ .

By Lemma 3.7, we have  $\mu_1(\Pi_r(go)) \asymp \mu_1(\Pi_{r+\delta}(go))$ . By the claim, for the direction “ $>_r$ ”, it suffices to show that any  $\xi \in \partial X$  is also evenly covered by shadows in  $\{\Pi_r(h \cdot t_U o) : h \in G_{U,L}, U \in \mathbb{U}_{r,n}(go)\}$ . In order to do so, we need the following property for a horoball  $U \in \mathbb{U}$  (cf. [11, Lemma 2.3]): let  $\gamma$  be a geodesic with endpoints  $\gamma_-, \gamma_+ \in \partial U$ . For any  $r > 0$ , there exists  $K = K(r)$  such that  $N_r(\partial U) \cap \gamma \subset N_K(\{\gamma_-, \gamma_+\})$ .

Let  $y, z_+$  be the corresponding entry and exit points of  $[1, \xi]$  in  $U$ . Recall that  $\partial U \subset N_M(G_U \cdot t_U o)$  by Lemma 3.16. Consider  $\xi \in \Pi_r(h \cdot t_U o)$ . There exists  $w \in [1, \xi]$  such that  $d(w, h \cdot t_U o) \leq r$  and then  $d(w, \partial U) \leq r + M$ . We have either  $d(y, w) \leq K$  or  $d(z_+, w) \leq K$  for  $K := K(r + M)$ . It is easy to see that  $d(y, w) \leq K$  cannot happen. See Figure 2. Thus,  $d(z_+, h \cdot t_U o) \leq K$ , and so there are at most finitely many  $\Pi_r(h \cdot t_U o)$  containing  $\xi$ . The proof is complete.  $\square$

**5.2. Distribution of horoballs in the partial cone  $\Omega_{r,\epsilon,R}(go)$ .** We now introduce some auxiliary sets to carry out a finer analysis of  $\mathbb{U}_{r,n}(go)$ . These sets divide  $\mathbb{U}_{r,n}(go)$  into a sequence of annulus sets according to the distance  $d(go, U)$ .

We first single out an exceptional set where  $o_U$  lies roughly “below”  $go$ . Precisely, let  $\mathbb{X}$  be the set of horoballs  $U \in \mathbb{U}_{r,n}(go)$  such that

$$(28) \quad d(o, o_U) - d(o, go) < r.$$

For  $i, \Delta \geq 0$  such that  $i + \Delta \leq n + 3r$ , consider the annulus set

$$\mathbb{U}_{r,n}(go, i, \Delta)$$

of  $U \in \mathbb{U}_{r,n}(go) \setminus \mathbb{X}$ , for which the following holds

$$(29) \quad |d(go, o_U) - i| \leq \Delta.$$

By (28) we have  $d(go, o_U) \geq r$  for  $U \in \mathbb{U}_{r,n}(go) \setminus \mathbb{X}$ .

*Lemma 5.4.* There exist  $r_0, \Delta_0 > 0$  such that the following hold for  $r \geq r_0, \Delta \geq \Delta_0$ :

- (1)  $\mathbb{U}_{r,n}(go) = \mathbb{X} \cup (\cup_{i \geq 0} \mathbb{U}_{r,n}(go, i, \Delta))$ ,
- (2)  $d(go, U) \leq 5r$  for any  $U \in \mathbb{X}$ .
- (3)  $\sharp \mathbb{U}_{r,n}(go, i, \Delta) <_{\Delta} \exp(i\delta_G)$  for any  $0 \leq i \leq n + 3r - \Delta$ .

*Remark.* We consider  $g = 1$ . Since  $\sharp \mathbb{X} < \infty$ , we see that  $\sharp \mathbb{U}_{r,n}(o, i, \Delta) \asymp \sharp H(o, i, \Delta)$  for  $i \leq n$ , where  $H(o, i, \Delta)$  (7) is the horoball growth function.

*Proof.* We first setup the basic inequalities used below. Given  $U \in \mathbb{U}_{r,n}(go)$ , let  $\xi \in \Pi_r^c(go)$  and  $z \in \gamma \cap U$  such that  $d(o, z) = d(o, go) + n$ , where  $\gamma := [o, \xi]$ . Choose  $x \in \gamma$  such that

$$(30) \quad d(o, x) = d(o, go).$$

As  $\xi \in \Pi_r(go)$ , we have

$$(31) \quad d(go, x) \leq 2r.$$

Let  $y \in [o, z]_{\gamma}$  be the entry point in  $U$  so that by Lemma 2.7, we have

$$(32) \quad d(y, o_U) \leq M.$$

As  $U \in \mathbb{U}$  is quasi-convex, let  $M$  also satisfy that  $[y, z]_{\gamma} \subset N_M(U)$ . Set  $r_0 = M$ .

We prove in order the statements (1), (2) and (3):

(1): Let  $U \in \mathbb{U}_{r,n}(go) \setminus \mathbb{X}$ . Then  $d(o, o_U) - d(o, go) \geq r$ . We claim that  $y \in [x, z]_\gamma$ . See Figure 2. Indeed, if not, assume  $y \in [o, x]_\gamma$  and then  $d(o, x) \geq d(o, y)$ . By (30) and (32), it follows that  $d(o, go) = d(o, x) \geq d(o, y) \geq d(o, o_U) - d(y, o_U) \geq d(o, o_U) - M$ . This gives a contradiction, since  $r > r_0 = M$ . Hence,  $y \in [x, z]_\gamma$  and then  $d(x, y) \leq n$  as  $d(x, z) = n$ . Then

$$d(go, o_U) \leq d(x, y) + d(y, o_U) + d(go, x) \leq d(x, y) + M + 2r \leq n + 3r.$$

So we have the upper bound in (29), and the statement (1) follows.

(2): Let  $U \in \mathbb{X}$  so that (28) holds. We see that  $y \in N_{2r}([o, x]_\gamma)$ . Indeed, if not, then  $y \in [x, z]_\gamma$  and  $d(x, y) > 2r$ . As  $r > r_0 = M$ , it follows that

$$\begin{aligned} d(o, x) &= d(o, y) - d(x, y) \\ &< d(o, y) - 2r < d(o, o_U) + M - 2r \\ &< d(o, o_U) - r, \end{aligned}$$

giving a contradiction with (28). Thus,  $y \in N_{2r}([o, x]_\gamma)$ .

As  $[y, z]_\gamma \subset N_M(U)$ , we obtain that  $d(x, U) \leq M + 2r$ . So by (31) we have  $d(go, U) \leq M + 4r \leq 5r$  for  $U \in \mathbb{X}$ . The statement (2) is proved.

(3): Let  $\Delta > \Delta_0 := M$  be given by Lemma 3.8 so that  $A(o, n, \Delta) <_\Delta \exp(\delta_G n)$ . For each  $U \in \mathbb{U}_{r,n}(go, i, \Delta)$ , there exists  $h_U \in G$  such that  $d(h_U o, o_U) < M$ . As  $\mathbb{U}$  is locally finite so that the number of  $U \in \mathbb{U}$  sharing a common  $h_U$  is uniformly finite, we have

$$\# \mathbb{U}_{r,n}(go, i, \Delta) <_M \# \{h_U : U \in \mathbb{U}_{r,n}(go, i, \Delta)\}.$$

It follows by (29) that

$$|d(go, h_U o) - i| \leq \Delta + M,$$

which yields

$$g^{-1}h_U \in A(o, i, M + \Delta).$$

By Lemma 3.8,

$$\# \mathbb{U}_{r,n}(go, i, \Delta) <_\Delta \exp(i\delta_G).$$

for  $i \geq 0$ . □

**5.3. The DOP condition.** The next observation, indicating the role of the DOP condition, is crucial in the proof of Theorem 1.7. We use the following notation

$$A_U(o, n, \Delta) := A(o, n, \Delta) \cap G_U$$

where  $n, \Delta \geq 1$  and  $o \in X, U \in \mathbb{U}$ .

*Lemma 5.5.* *The group  $G$  satisfies the DOP condition if and only if the following holds*

$$\sum_{p \in G_U} d(v, pv) \cdot \exp(-\delta_G d(v, pv)) \asymp_\Delta \sum_{j \geq 0} \sum_{m \geq j} \# A_U(v, m, \Delta) \cdot \exp(-\delta_G m) < \infty,$$

for any  $U \in \mathbb{U}, v \in \partial U$  and  $\Delta > 1$ .

*Proof.* Observe that

$$\sum_{p \in G_U} d(v, pv) \cdot \exp(-\delta_G d(v, pv)) \asymp_\Delta \sum_{m \geq 0} m \cdot \# A_U(v, m, \Delta) \cdot \exp(-\delta_G m),$$

for any  $\Delta \geq 1$ . Indeed, the “ $\leq$ ” direction is trivial. For the “ $>$ ” direction, it suffices to notice that for  $p \in G_U$  the point  $pv$  lies in at most  $2\Delta + 1$  sets from  $\{A_U(v, n, \Delta) : n \in \mathbb{N}\}$ .

The conclusion follows from a re-arrangement of the above series. □

Let  $\Delta_0 > 0$  be given by Lemma 5.4. The following notation will be useful by Lemma 5.5: for any  $V \in \mathbb{U}$  and  $L > 0$ ,

$$(33) \quad \Phi_{V,L}(n) = \sum_{j \geq L}^{j \leq n+L+3r-\Delta_0} \sum_{m \geq j} \# A_V(o_V, m, \Delta_0) \cdot \exp(-\delta_G m).$$

We recommend the reader to read first Lemma 5.7, where the following technical lemma is used.

*Lemma 5.6. Let  $r > 0$  given by Lemma 3.18. For any  $n, L > 0$  consider the following series*

$$(34) \quad \Theta_L(go, n) = \sum_{i \geq 0}^{i \leq n+3r-\Delta_0} \sum_{U \in \mathbb{U}_{r,n}(go, i, \Delta_0)} \left( \exp(-i\delta_G) \sum_{m \geq n-i+L} \# A_U(o_U, m, \Delta_0) \cdot \exp(-\delta_G m) \right).$$

Then the following hold:

- (1)  $\Theta_L(go, n) <_{r, \Delta_0} \Phi_{V,L}(n)$  for any  $V \in \mathbb{U}$ .
- (2) If the horoball growth function is purely exponential, then

$$\Theta_L(o, n) >_{r, \Delta_0} \Phi_{V,L}(n).$$

*Proof.* Let  $K$  be the number of  $G$ -orbits in  $\mathbb{U}$ . Then  $K < \infty$  by a theorem of Tukia [19]. Let  $\{U_k \in \mathbb{U} : 1 \leq k \leq K\}$  be a choice of representatives for each  $G$ -orbit in  $\mathbb{U}$  such that  $d(o, U_k)$  is minimal among the corresponding  $G$ -orbit of  $U_k$ . Then  $d(o, U_k) \leq \|X \setminus \mathcal{U}/G\| \leq M$  for  $1 \leq k \leq K$ . This implies that any two series as (34) which are summing up over  $U \in \mathbb{U}_{r,n}(go)$  in different  $G$ -orbits are bi-Lipschitz, with a Lipschitz constant depending on  $M$ . Without loss of generality, we can assume that (34) is summing up over  $U \in \mathbb{U}_{r,n}(go)$  in the same  $G$ -orbit, up a constant depending on  $M, K$ .

Let  $U, U' \in \mathbb{U}$  such that  $U' = gU$  for some  $g \in G$ . Then there exists  $C_1 > 0$  such that

$$(35) \quad C_1^{-1} \# A_{U'}(x', m, \Delta_0) \leq \# A_U(x, m, \Delta_0) \leq C_1 \# A_{U'}(x', m, \Delta_0)$$

for any  $x \in \partial U$  and  $x' \in \partial U'$ . Indeed, this follows from the fact that  $G_U$  acts co-compactly on  $\partial U$ . The point  $x$  is mapped by  $g \in G$  into a uniform neighborhood of  $G_U o$ . And  $\# A_U(x, m, \Delta) = \# A_U(hx, m, \Delta)$  for any  $h \in G_U$ . So (35) follows.

By Lemma 5.4 (3), there exists  $C_2 = C_2(r, \Delta_0) > 0$  such that

$$(36) \quad \# \mathbb{U}_{r,n}(go, i, \Delta_0) \leq C_2 \exp(i\delta_G).$$

for any  $i > 0$  such that  $i + \Delta_0 \leq n + 3r$ .

By the first two paragraphs, we fix  $V \in \mathbb{U}_{r,n}(go)$  and consider the sum (34) over  $G \cdot V \subset \mathbb{U}_{r,n}(go)$ . By (35) and (36), we see that

$$(37) \quad \Theta_L(go, n) <_{r, \Delta_0} \sum_{i \geq 0}^{i \leq n+3r-\Delta_0} \left( \sum_{m \geq n-i+L} \# A_V(o_V, m, \Delta_0) \cdot \exp(-\delta_G m) \right)$$

which further yields:

$$\Theta_L(go, n) <_{r, \Delta_0} \sum_{j \geq L}^{j \leq n+L+3r-\Delta_0} \sum_{m \geq j} \# A_V(o_V, m, \Delta_0) \cdot \exp(-\delta_G m)$$

for any  $n \geq 0$ . The statement (1) is proved.

Assume now that the horoball growth function is purely exponential. By Lemma 4.1, there exists a constant, still denoted by  $C_2$ , such that

$$(38) \quad C_2^{-1} \exp(i\delta_G) \leq \# H_V(o, i, \Delta_0).$$

for any  $i > 0$  such that  $i + \Delta_0 \leq n + 3r$ . By the remark after Lemma 5.4,  $H_V(o, i, \Delta_0) \asymp \mathbb{U}_{r,n}(o, i, \Delta_0)$ . Similarly, by making use of (35) and (38), we obtain for  $g = 1$ :

$$\Theta_L(o, n) >_{r, \Delta_0} \Phi_{V,L}(n),$$

proving the statement (2).  $\square$

**5.4. Horospherical shadows.** The key is to analyze the horospherical shadows.

*Lemma 5.7.* *We fix  $g \in G$  and  $V \in \mathbb{U}$ . Let  $\Phi_{V,L}(n)$  be defined in (33). For any  $r > M, L > 0$  the following holds.*

$$(39) \quad \sum_{h \in G_{U,L}}^{U \in \mathbb{U}_{r,n}(go)} \exp(-\delta_G d(o, h \cdot t_U o)) <_r \exp(-\delta_G d(o, go)) \cdot \Phi_{V,L}(n)$$

for any  $n \gg 0$ . If  $G$  has purely exponential growth for horoballs, then

$$(40) \quad \sum_{h \in G_{U,L}}^{U \in \mathbb{U}_{r,n}(o)} \exp(-\delta_G d(o, h \cdot t_U o)) >_r \Phi_{V,L}(n)$$

for any  $n \gg 0$ .

*Proof.* We follow the notations given at the beginning of proof of Lemma 5.4. Moreover, let  $z_+$  be the exit point of  $\gamma$  in  $U$ . By Lemma 3.16, there exist  $h \in G_U, t_U \in G$  such that

$$(41) \quad d(h \cdot t_U o, z_+) \leq M, \quad d(t_U o, o_U) \leq M$$

We assumed  $h \in G_{U,L}$  so that  $d(z, z_+) > L$  and then  $d(x, z_+) \geq n + L$ .

Since  $\mathbb{U}_{r,n}(go) = \mathbb{X} \cup (\cup_{i \geq 0} \mathbb{U}_{r,n}(go, i, \Delta_0))$  by Lemma 5.4, we examine the following two cases. Note that  $r > M$  and (30) (31) (32) will be used often in what follows.

**Case 1.** Consider  $U \in \mathbb{X}$ . Since  $d(go, U) < 5r$ ,  $\mathbb{X}$  is a finite set. By (41) and (31) we have

$$d(go, h t_U o) \sim_{3r} d(x, z_+),$$

where  $\sim_{3r}$  denotes the quantities at both sides is equal up to an additive constant  $3r$ . So we obtain

$$d(o, z_+) = d(o, x) + d(x, z_+) \sim_{3r} d(o, go) + d(go, h \cdot t_U o)$$

and then

$$(42) \quad d(o, h \cdot t_U o) \sim_M d(o, z_+) \sim_{4r} d(o, go) + d(go, h \cdot t_U o)$$

By Lemma 3.17 we have

$$(43) \quad \sum_{h \in G_U} \exp(-\delta_G d(z, hw)) < \infty$$

where  $v, w \in N_{5r}(U)$ . As  $d(go, U) < 5r$  and  $d(t_U o, U) \leq d(t_U o, o_U) \leq M \leq r$ , we apply Lemma 3.17 for  $go, t_U o \in N_{5r}(U)$  so that (43) holds for  $v := go, w := t_U o$ .

Note that we have  $d(go, h \cdot t_U o) \geq d(x, z_+) - 3r \geq n + L - 3r$ . By (42) and (43), we have

$$\begin{aligned}
 (44) \quad & \sum_{U \in \mathbb{X}} \exp(-\delta_G d(o, h \cdot t_U o)) \\
 & \asymp \sum_{U \in \mathbb{X}} \exp(-\delta_G d(o, go)) \cdot \exp(-\delta_G d(go, h \cdot t_U o)) \\
 & \asymp \exp(-\delta_G d(o, go)) \cdot \sum_{h \in G_U}^{d(go, h \cdot t_U o) \geq n+L-3r} \exp(-\delta_G d(z, hw)),
 \end{aligned}$$

where we used that  $\mathbb{X}$  is finite set. Since the series (43) is convergent, we see that for  $n \gg 0$ , the series (44) can be arbitrarily small and thus can be ignored. So it suffices to consider the **Case (2)**.

**Case 2.** Consider  $U \in \mathbb{U}_{r,n}(go, i, \Delta_0)$  for  $i \geq 0$ . By the proof of Lemma 5.4 (1), we see that  $y \in [x, z]_\gamma$ , and  $d(z, z_+) = d(y, z_+) - d(y, z)$ . By (32) and (41), this implies

$$d(z, z_+) = d(z_+, y) - d(y, z) \sim_{3M} d(o_U, h \cdot t_U o) - d(o_U, z)$$

which yields

$$\begin{aligned}
 (45) \quad d(o, h \cdot t_U o) & \sim_M d(o, z_+) = d(o, z) + d(z, z_+) \\
 & \sim_{4M} (d(o, go) + n) + d(o_U, h \cdot t_U o) - d(o_U, z).
 \end{aligned}$$

Recall that  $r > M$  and  $d(y, o_U) \leq M$ . Since  $d(x, z) = n$  and  $y \in [x, z]_\gamma$ , it follows

$$n + 4r > d(go, o_U) + d(o_U, z) > n - 2r.$$

Since  $U \in \mathbb{U}_{r,n}(go, i, \Delta_0)$ , it follows by (29) that

$$n - i + \Delta_0 + 4r > d(o_U, z) > n - i - \Delta_0 - 2r.$$

By (45), we have

$$\begin{aligned}
 (46) \quad & \exp(-\delta_G d(o, h \cdot t_U o)) \\
 & \asymp_{r, \Delta_0} \exp(-\delta_G (d(o, go) + i)) \cdot \exp(-\delta_G d(o_U, h \cdot t_U o)).
 \end{aligned}$$

On the other hand, we estimate

$$\begin{aligned}
 d(o_U, h \cdot t_U o) & \sim_{2M} d(y, z_+) = d(y, z) + d(z, z_+) \\
 & \sim_{3M} d(o_U, z) + d(z, z_+).
 \end{aligned}$$

Recall that  $d(z, z_+) > L$  and so

$$d(o_U, h \cdot t_U o) \geq n - i - \Delta_0 + L - 5r.$$

Since  $d(t_U o, o_U) \leq M$ , we replace  $t_U o$  by  $o_U$  to get

$$(47) \quad \sum_{h \in G_{U,L}} \exp(-\delta_G d(o_U, h \cdot t_U o)) \asymp_{r, \Delta_0} \sum_{m \geq n-i+L} \# A_U(o_U, m, \Delta_0) \cdot \exp(-\delta_G m).$$

By (46) and (47), we obtain for a fixed  $i$ :

$$\begin{aligned}
 (48) \quad & \sum_{\substack{h \in G_{U,L} \\ U \in \mathbb{U}_{r,n}(go, i, \Delta_0)}} \exp(-\delta_G d(o, h \cdot t_U o)) \asymp_{r, \Delta_0} \exp(-\delta_G (d(o, go) + i\Delta_0)) \cdot \\
 & \left( \sum_{U \in \mathbb{U}_{r,n}(go, i, \Delta_0)} \sum_{m \geq n-i+L} \# A_U(o_U, m, \Delta_0) \cdot \exp(-\delta_G m) \right)
 \end{aligned}$$

We sum up (48) for  $0 \leq i \leq n + 3r - \Delta_0$ . By Lemma 5.6, we have that (39) and (40) follow separately from the statements (34) and (33). The lemma is proved.  $\square$



*Corollary 5.8.* Assume that the group  $G$  satisfies the DOP condition. Let  $r > 0$  given by Lemma 3.18. For any  $\varepsilon > 0$  there exists  $L = L(\varepsilon, r) > 0$  such that

$$(49) \quad \sum_{\substack{U \in \mathbb{U}_{r,n}(go) \\ h \in G_{U,L}}} \exp(-\delta_G d(o, h \cdot t_U o)) < \varepsilon \cdot \exp(-\delta_G d(o, go)).$$

*Proof.* This follows from Lemmas 5.5 and 5.7.  $\square$

## 6. PROOF OF THEOREM 1.7

**1. The DOP condition  $\Rightarrow$  purely exponential orbit growth in cones:** Let  $r, \varepsilon, R, C_1, C_2$  be given by Lemma 3.18 so that

$$C_1 \cdot \exp(-\delta_G d(o, go)) < \mu_1(\Pi_{r,\varepsilon,R}^c(go)) \leq \mu_1(\Pi(go)) \leq C_2 \cdot \exp(-\delta_G d(o, go)).$$

for any  $g \in G$ .

For  $r, C_1/2 > 0$ , there exists  $L = L(C_1/2, r) > 0$  by Corollary 5.8 such that

$$(50) \quad \sum_{\substack{U \in \mathbb{U}_{r,n}(go) \\ h \in G_{U,L}}} \exp(-\delta_G d(o, h \cdot t_U o)) < C_1/(2C_2) \cdot \exp(-\delta_G d(o, go)).$$

By the forumule (26) of Lemma 5.3, there exist  $R' = R'(\varepsilon, R), \Delta = \Delta(L) > 0$  such that the following holds

$$(51) \quad \begin{aligned} C_1 \cdot \exp(-\delta_G d(o, go)) &\leq C_2 \sum_{\substack{h \in \Omega_{r+\delta,\varepsilon,R'}(go,n,\Delta) \\ U \in \mathbb{U}_{r,n}(go) \\ h \in G_{U,L}}} \exp(-\delta_G d(o, ho)) \\ &\quad + C_2 \sum_{\substack{U \in \mathbb{U}_{r,n}(go) \\ h \in G_{U,L}}} \exp(-\delta_G d(o, h \cdot t_U o)). \end{aligned}$$

for  $g \in G$  and  $n \gg 0$ .

Note that  $d(o, ho) > d(o, go) + n - \Delta$  for any  $h \in \Omega_{r+\delta,\varepsilon,R'}(go, n, \Delta)$ . By (50) and (51), we have the following

$$C_1/(2C_2) < \# \Omega_{r+\delta,\varepsilon,R'}(go, n, \Delta) \cdot \exp(-\delta_G n) \cdot \exp(\Delta),$$

yielding

$$\# \Omega_{r+\delta,\varepsilon,R'}(go, n, \Delta) >_{r,\Delta} \exp(-\delta_G n).$$

Thus,  $G$  has purely exponential growth in cones.

**2. Purely exponential orbit growth  $\Rightarrow$  Purely exponential horoball growth:** This is already proved in Lemma 4.1.

**3. Purely exponential horoball growth  $\Rightarrow$  the DOP condition:** We apply previous results to  $g = 1$ . Fix  $L > 0$  and  $V \in \mathbb{U}$ . Assume that we have purely exponential growth for horoballs. By Lemmas 5.3 and 5.7 we have

$$\mu_1(\Pi_r(o)) >_r \sum_{\substack{U \in \mathbb{U}_{r,n}(o) \\ h \in G_{U,L}}} \exp(-\delta_G d(o, h \cdot t_U o)) >_r \Phi_{V,L}(n)$$

for any  $n \gg 0$ . The DOP condition follows from Lemma 5.5.

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